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COMPARISON OF THE INFLUENCES OF INITIAL ERRORS AND MODEL PARAMETER ERRORS ON PREDICTABILITY OF NUMERICAL FORECAST

DING Rui-Qiang, LI Jian-Ping

State Key Laboratory of Numerical Modeling for Atmospheric Sciences and Geophysical Fluid Dynamics, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, *China*

Abstract Based on the nonlinear local Lyapunov exponent (NLLE) approach introduced by the authors recently, the influences of initial errors and parameter errors on the predictability of the Lorenz system are studied. The major results are summarized as follows: (i) When only initial errors or only parameter errors exist in the Lorenz system, the error growth and predictability limit are comparable between two kinds of predictability problems. This result holds basically for the wide range of parameter space of the Lorenz system. (ii) When initial errors and parameter errors exist together in the Lorenz system, their effects of on the predictability of the Lorenz system mainly depend on their relative magnitudes. When the magnitude of initial errors is much greater than that of parameter errors, the predictability limit of the Lorenz system is mainly determined by initial errors. On the contrary, when the magnitude of parameter errors is much greater than that of initial errors, the predictability limit is mainly determined by parameter errors. When the magnitude of initial errors is close to that of parameter errors, initial errors and parameter errors contribute together to the predictability limit of the Lorenz system. These results indicate that when numerical weather forecast is made, we should pay great attention to the determination of model parameters, as well we to the determination of initial conditions.

Key words Numerical weather forecast, Predictability, Initial error, Parameter error, Lorenz system

1 INTRODUCTION

Lorenz[1] called the initial value problem as predictability of the first kind, which focuses on the effect of the uncertainties in the initial state of the numerical model on the predictability. There have been a lot of studies done by using numerical models to study the effect of initial errors on the predictability[2~6]. These studies all assume that the numerical model is perfect (i.e., the model equations and model parameters are accurate) and the prediction error mainly comes from the initial errors. However, in real situations, even though the model equations are accurate, model parameters (such as the coefficients in model equations) need to be given besides the intimal states in order to make prediction. These parameters are usually determined by observational experiments. Inevitably, there exist some errors in model parameters. The parameter errors will bring the uncertainties to the prediction results, which leads to the predictability problem of parameter errors.

Chu[7] compared the effects of parameter errors in the Lorenz system on the predictability with that of initial errors. He found that the effect of parameter errors on the predictability is comparable to that of initial errors, and it is therefore important to prepare the accurate model parameters. Bian et al.[8] also investigated the predictability problem of initial errors and parameter errors in the Lorenz system. They studied the error growth when only initial errors or only parameter errors exist in the Lorenz system. In addition, different from the study of Chu[7], Bian et al.[8] further explored the error growth when initial errors and parameter errors exist together in the Lorenz system. They concluded that it is not necessary to consider the effect of small parameter errors when initial errors and parameter errors exist together in the Lorenz system. These studies are important for us to understand the relative importance of initial errors and parameter errors in the predictability problem. However, previous studies are found to have some limitations. Firstly, they are all based on a single initial value, which only reflects the effect of initial error sand parameter error on the predictability

E-mail: drq@mail.iap.ac.cn

in the local phase space. For the average effect of initial errors and parameter errors on the predictability of the whole system, there are no studies to be done. Secondly, previous studies only estimate qualitatively the predictability under the condition of the initial errors and parameter errors given. They do not determine quantitatively the predictability for the given initial errors and parameter errors.

In view of the limitations of linear error growth dynamics in the predictability study, Ding and Li[9] introduced a definition of the nonlinear local Lyapunov exponent (NLLE) recently. The NLLE measures the short-term growth rate of initial errors of nonlinear dynamical models without linearizing the governing equations. With the NLLE and its derivatives, the limit of dynamic predictability in large classes of chaotic systems can be efficiently and quantitatively determined[10~12]. In this paper, we shall compare the predictability problems of initial errors and parameter errors using the NLLE approach. Aiming at the limitations of previous studies, we take the Lorenz system as an example and focus on investigating the following predictability problems: (a) Under the condition of the system parameter given, comparing the changes of predictability limit with initial errors and parameter errors of different magnitudes; (b) Under the condition of the initial errors and parameter errors given, comparing the changes of predictability limit with the parameter range; and (c) Under the condition of the initial errors and parameters error existing together, exploring the relative importance of the initial errors and parameter errors.

2 NLLE AND QUANTIFICATION OF THE PREDICTABILITY LIMIT

For an n-dimensional nonlinear dynamical system, the solutions of full error evolution equations without any approximation can be obtained by numerically integrating it along the reference solution from $t = t_0$ to $t_0 + \tau$:

$$\boldsymbol{\delta}(t_0+\tau) = \boldsymbol{\tau}(\boldsymbol{x}(t_0), \boldsymbol{\tau}(t_0), \tau)\boldsymbol{\delta}(t_0), \tag{1}$$

where $\boldsymbol{\eta}(\boldsymbol{x}(t_0), \boldsymbol{\tau}(t_0), \tau)$ is the nonlinear propagator, $\boldsymbol{x}(t_0)$ initial state, $\boldsymbol{\delta}(t_0)$ initial error, and τ evolution time. Then the nonlinear local Lyapunov exponent (NLLE) is defined as

$$\lambda(\boldsymbol{x}(t_0), \boldsymbol{\delta}(t_0), \tau) = \frac{1}{\tau}\ln\frac{\|\boldsymbol{\delta}(t_0+\tau)\|}{\|\boldsymbol{\delta}(t_0)\|}, \tag{2}$$

where $\lambda(\boldsymbol{x}(t_0), \boldsymbol{\delta}(t_0), \tau)$ depends in general on the initial state in phase space $\boldsymbol{x}(t_0)$, the initial error $\boldsymbol{\delta}(t_0)$, and time τ. The NLLE is different from the finite-time Lyapunov exponent or local Lyapunov exponent defined previously based on the linear error dynamics[13~15], which only depends on the initial state $\boldsymbol{x}(t_0)$ and the time τ, and does not depend on the initial error $\boldsymbol{\delta}(t_0)$. For notational simplicity, let the norm of error in phase space at time t be $\delta(t) = \|\boldsymbol{\delta}(t)\|$. To study the average dynamical behavior of the whole nonlinear system, the mean NLLE over attractor is given by

$$\overline{\lambda}(\boldsymbol{\delta}(t_0), \tau) = \langle\lambda(\boldsymbol{x}(t_0), \boldsymbol{\delta}(t_0), \tau)\rangle_N = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\tau}\ln\frac{\delta_i(t_0+\tau)}{\delta_i(t_0)}, \tag{3}$$

where $\langle\rangle_N$ denotes the ensemble average of samples of large enough size $N(N \to \infty)$. The mean relative growth of initial error (RGIE) $\overline{E}(\boldsymbol{\delta}(t_0), \tau)$ can be obtained by

$$\overline{E}(\boldsymbol{\delta}(t_0), \tau) = \exp(\overline{\lambda}(\boldsymbol{\delta}(t_0), \tau)\tau). \tag{4}$$

From Eqs.(2), (3) and (4), we get

$$\overline{E}(\boldsymbol{\delta}(t_0), \tau) = \exp\left(\frac{1}{N}\sum_{i=1}^{N}\ln\frac{\delta_i(t_0+\tau)}{\delta_i(t_0)}\right). \tag{5}$$

For the same initial error $\delta(t_0)$, we have

$$\overline{E}(\boldsymbol{\delta}(t_0),\tau)=\frac{\left(\prod_{i=1}^{N}\delta_i(t_0+\tau)\right)^{\frac{1}{N}}}{\delta(t_0)}. \tag{6}$$

For chaotic systems, as $\tau\to\infty$, $\delta_1(t_0+\tau),\delta_2(t_0+\tau),\cdots,\delta_N(t_0+\tau)$ will follow an independent identically distribution. Using the theorem proved by Ding and Li[9], then we obtain

$$\overline{E}(\boldsymbol{\delta}(t_0),\tau)\xrightarrow{P}c(N\to\infty), \tag{7}$$

where $\xrightarrow{P}$ denotes the convergence in probability and c can be considered as the theoretical saturation level of $\overline{E}(\boldsymbol{\delta}(t_0),\tau)$. Using the theoretical saturation level, the limit of dynamic predictability can be quantitatively determined (here the limit of dynamic predictability corresponds to an average time scale $T_{\rm p}$, beyond which the prediction will become meaningless owing to the propagation of initial errors over the entire attractor).

3 EXPERIMENTS

The Lorenz system[16] is

$$\begin{cases}\dot{X}=-\sigma X+\sigma Y,\\ \dot{Y}=rX-Y-XZ,\\ \dot{Z}=XY-bZ,\end{cases} \tag{8}$$

where σ is called the Prandial number, r is called the Rayleigh number (hereafter σ replaced by Pr and r replaced by Ra), and b is the parameter number. Assume that the accurate initial states of the Lorenz system are $X(t_0),Y(t_0)$ and $Z(t_0)$, the accurate parameters of the Lorenz system are Pr,Ra and b, the observed initial states are $X'(t_0),Y'(t_0)$ and $Z'(t_0)$, and the observed parameters are Pr',Ra' and b'. The following three kinds of predictability problems are studied.

(1) Sensitivity to the initial error (i.e., the error only exists in the initial states and control parameters are accurate)

$$\begin{aligned}&X'(t_0)=X(t_0),\quad Y'(t_0)=Y(t_0),\quad Z'(t_0)=Z(t_0)(1+\varepsilon),\\ &Pr'=Pr,\quad Ra'=Ra,\quad b'=b,\end{aligned}$$

where ε is the small error introduced in the initial states.

(2) Sensitivity to the parameter error (i.e., the error only exists in the control parameters and initial states are accurate)

$$\begin{aligned}&X'(t_0)=X(t_0),\quad Y'(t_0)=Y(t_0),\quad Z'(t_0)=Z(t_0),\\ &Pr'=Pr,\quad Ra'=Ra(1+\varepsilon),\quad b'=b,\end{aligned}$$

where ε is the small error introduced in the control parameters.

(3) Sensitivity to both the initial error and parameter error (i.e., the initial error and parameter error exist together in the Lorenz system)

$$\begin{aligned}&X'(t_0)=X(t_0),\quad Y'(t_0)=Y(t_0),\quad Z'(t_0)=Z(t_0)(1+\varepsilon),\\ &Pr'=Pr,\quad Ra'=Ra(1+\varepsilon),\quad b'=b.\end{aligned}$$

The error growth due to the initial error, parameter error, or both can be obtained from the time evolution of the difference between the solutions of sensitivity experiments and the accurate solution of the Lorenz system. In Refs.[9∼12], the NLLE approach is mainly used to study the predictability problem of the initial value. The general idea of the NLLE approach is to investigate the nonlinear growth of the initial error under the assumption of the initial error existing in the initial states. Therefore, the NLLE approach can not be directly used to study the predictability problem of the parameter error. But if the parameter Ra is considered as another variable of the Lorenz system besides the variables X, Y, Z, then the predictability problem of the parameter error can change into the predictability problem of the initial error. In this way the NLLE approach can be used to study the error growth due to the parameter error. In addition, according to the introduction of NLLE approach in the Section 2, we take the ensemble average of the NLLE over a great number of initial states on the same orbit obtained by long time integration of the Lorenz system. Then the error growth curves that reflect the global feature of the whole Lorenz system can be obtained. Based on these curves, the predictability limit of the Lorenz system can be quantitatively determined.

4 RESULTS

Three kinds of predictability problems to be studied have been listed in the third section. In the following parts, the experimental results are given.

4.1 Comparison of Dependencies of Predictability Limit on the Magnitudes of Initial Error and Parameter Error

For the given control parameters of the Lorenz system $Pr = 10, Ra = 28$, and $b = 8/3$, Fig. 1(a,b) show the dependence of the mean RGIE on the magnitudes of the initial error and parameter error, respectively. It is indicated that for the initial error or for the parameter error, the error growth curves of the Lorenz system are similar. In two cases, the mean RGIE initially increases and finally reaches saturation as the time increases. The time that error growth reaches the saturation increases as the initial error decreases. The same thing happens to the case of the parameter error. If the predictability limit is defined as the time at which the error reaches 99% of its saturation level, it is found that the predictability limit decreases as the initial error or parameter error increases (Fig. 2). For the initial error and parameter error of same magnitude, the Lorenz system has almost similar predictability limit. It is shown that the sensitivity of the Lorenz system to the initial error and parameter error is similar. From this point of view, estimating accurately the initial states is as important as estimating accurately the control parameters of the system in order to improve the prediction.

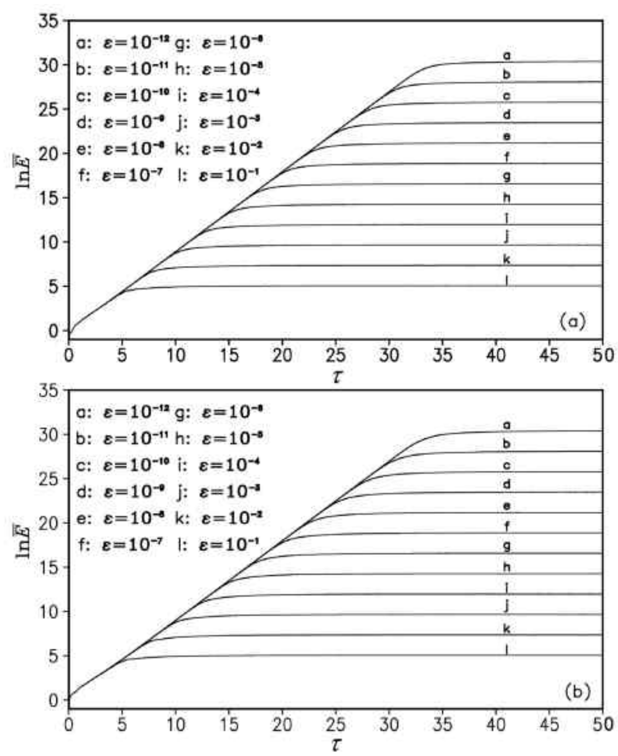

Fig. 1 For the initial error (a) and parameter error (b) of different magnitudes, the mean relative growth of initial error $\overline{E}(\boldsymbol{\delta}(t_0), \tau)$ of Lorenz system as a function of time τ

4.2 Comparison of Dependencies of Predictability Limit on Parameter Range When Initial Error and Parameter Error are Given

As shown in Fig. 3, when the magnitude of errors is given as $\varepsilon = 10^{-6}$, the changes of predictability limit due to the initial error with the parameter Ra are similar to those of predictability limit due to the parameter error. The predictability limit due to the initial error and parameter error both show a decreasing trend as Ra increases. Whether for the initial error or for the parameter error, the predictability limit varies gently with the parameter Ra. A little change of the parameter Ra does not lead to the big change of predictability limit. This implies that a little change of the control parameter would not cause an abrupt shift of dynamic properties of the Lorenz system (i.e., control parameter is not near the critical point). For the most values of parameter range, the predictability limit due to the initial error is totally close to the one due to the parameter error. Only for several individual values of parameter range, there are big differences between the predictability limits due to the initial error and that due to the parameter error. It indicates that the conclusions obtained under the condition of the control parameter given show no obvious change with parameter Ra. It also shows the reasonability under the condition of the control parameter given to investigate the predictability problems of the initial error and parameter error.

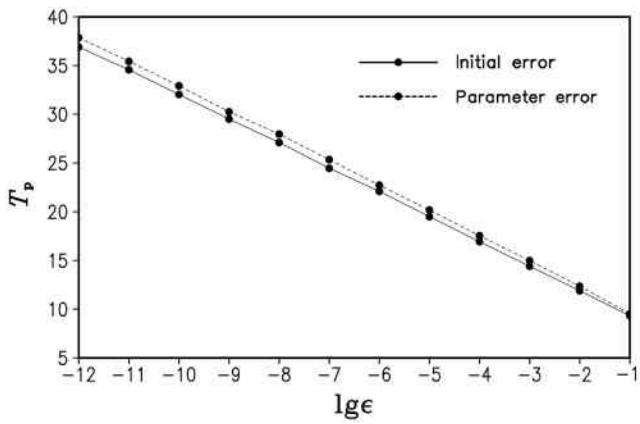

Fig. 2 The predictability limit T_{p} of Lorenz system as a function of initial error and parameter error of different magnitudes respectively

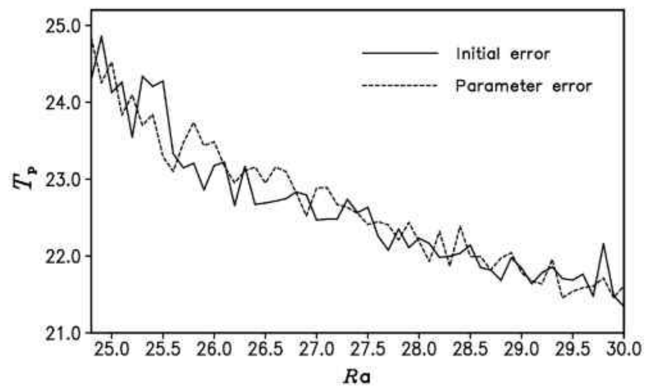

Fig. 3 With the magnitude of initial error and parameter error both equal to 10^{-6}, the predictability limit T_{p} of Lorenz system as a function of boundary forcing parameter Ra

4.3 Comparison of the Relative Importance of the Initial Error and Parameter Error to Predictability Limit

In real situations, the initial error and parameter error often exist together. Which one has the more important effect on the predictability limit? Can the effect of the parameter error be neglected since the existence of the initial error is inevitable? It is seen from Fig. 4 that when the parameter error varies from 10^{-12} to 10^{-7}, which is much smaller than the value 10^{-6} of the initial error, the predictability limit of the Lorenz system does not change with the parameter error. The predictability limit is similar to the one with the magnitude of the initial error 10^{-6} and no parameter error. When the parameter error varies from 10^{-5} to 10^{-3}, which is much greater than the value 10^{-6} of the initial error, the predictability limit of the Lorenz system mainly depends on the the parameter error. The predictability limit is similar to that with the parameter error and without the initial error. When the initial error and parameter error both are 10^{-6}, the predictability limit of the Lorenz system is higher than the one when only the initial error or only the parameter error exists. Fig. 5 shows the dependencies of the predictability limit of the Lorenz system on the magnitudes of the initial error when the parameter error is given as 10^{-6}. The results of Fig. 5 are found to be very similar to those of Fig. 4.

5 CONCLUSIONS

In this paper, the NLLE approach is used to compare the predictability problems of the initial error and parameter error. Three kinds of predictability problems are studied separately. The main conclusions are summarized as follows:

(1) When only the initial error or only the parameter error exists in the Lorenz system, the error growth

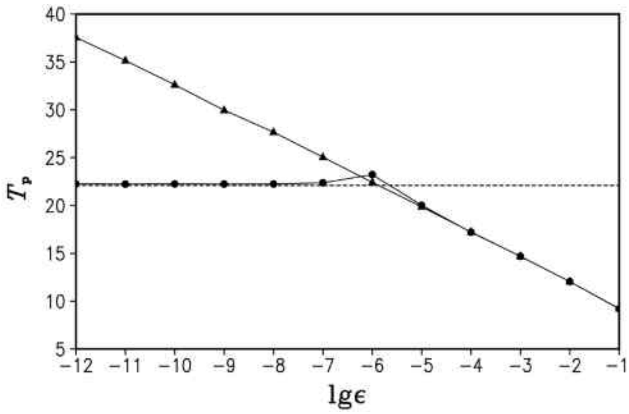

Fig. 4 For a constant initial error of 10^{-6}, the predictability limit $T_{\rm p}$ of Lorenz system as a function of parameter error of different magnitudes (closed circle)

The predictability limit $T_{\rm p}$ of Lorenz system as a function of parameter error without initial error is denoted by closed triangle. Dashed line denotes the predictability limit of Lorenz system with the initial error equal to 10-6 and no parameter error.

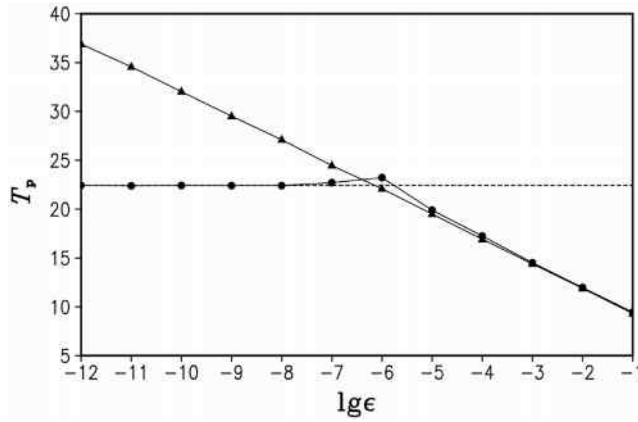

Fig. 5 For a constant parameter error of 10^{-6}, the predictability limit of Lorenz system as a function of initial error of different magnitudes (closed circle)

The predictability limit of Lorenz system as a function of initial error without parameter error is denoted by closed triangle. Dashed line denotes the predictability limit of Lorenz system with the parameter error equal to 10^{-6} and no initial error.

curves of the Lorenz system are similar for two cases. The mean relative error in two cases both initially increases and finally reaches saturation as the time increases. The predictability limit decreases as the initial error or the parameter error increases. For the initial error and parameter error of same magnitude, the Lorenz system has almost similar predictability limit. From this point of view, estimating accurately the initial states is as important as estimating accurately the control parameters of the system in order to improve prediction.

(2) When the initial error and parameter error are given, the trend changes of predictability limit due to the initial error with the parameter Ra are similar to those of predictability limit due to the parameter error. For the most values of parameter range, the predictability limit due to the initial error is overally close to the one due to the parameter error. It is indicated that the conclusions obtained under the condition of the control parameter given show no obvious change with parameter Ra.

(3) When the initial error and parameter error exist together in the Lorenz system, the effects of the initial error and parameter error on the predictability of the Lorenz system mainly depend on their relative magnitudes. When the magnitude of the initial error is much greater than that of the parameter error, the predictability limit of the Lorenz system is mainly determined by the initial error. On the contrary, when the magnitude of the parameter error is much greater than that of the initial error, the predictability limit is mainly determined by the parameter error. When the magnitude of the initial error is close to that of the parameter error, the initial error and parameter error contribute jointly to the predictability limit of the Lorenz system. In this case the effect of the parameter error on the predictability must be considered with the same importance as the initial error.

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